

L-embedded Banach spaces and measure topology

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Abstract

An L-embedded Banach space is a Banach space which is complemented in its bidual such that the norm is additive between the two complementary parts. On such spaces we define a topology, called an abstract measure topology, which by known results coincides with the usual measure topology on preduals of finite von Neumann algebras (like $L^1([0, 1])$). Though not numerous, the known properties of this topology suffice to generalize several results on subspaces of $L^1([0, 1])$ to subspaces of arbitrary L-embedded spaces.

§1 Introduction

This article continues the investigations made in [23, 24] on asymptotically isometric copies of l^1 in preduals of von Neumann algebras and in L-embedded Banach spaces. (For definitions see below.) In [24] it has been proved that, roughly speaking, in the predual of a finite von Neumann algebra the only non-trivial bounded sequences that converge to 0 with respect to the measure topology are essentially those that span l^1 asymptotically; for $L^1(\mu)$, μ a finite measure, this characterization has been known for quite a time [15, Th. 2], [25, Th. 3, Rem. 6bis].

From the point of view of Banach space theory, L-embedded Banach spaces provide a natural frame for preduals of von Neumann algebras. So the starting point of this paper is on the one hand the definition of an abstract measure topology, Definition 3, patterned after the just mentioned characterization and on the other hand the easy but important observation, Theorem 4, that every L-embedded space admits such a topology. Although this topology does not come out easily with its properties - at the time of this writing it is not clear whether it is Hausdorff let alone metrizable or whether addition is continuous - it allows to generalize several results on subspaces of $L^1(\mu)$ to subspaces of arbitrary L-embedded spaces. Thus section §4 of the present paper is titled "Section IV.3 of [13] (partly) revisited". For example, Theorem 10 generalizes a theorem of Buhvalov-Lozanovskii which describes the link between L-embeddedness and measure topology for subspaces Y of $L^1(\mu)$, μ finite: Y is L-embedded if and only if its unit ball is closed in measure. (Note in passing that this criterion involves only the space Y itself, not its bidual.) Moreover, as a consequence of this, the closedness in measure of the unit ball of Y is a weak substitute for compactness which could be called "convex sequential compactness", see Corollary 9. We also reprove a result of Godefroy and Li concerning a criterion for L-embedded subspaces which are duals of M-embedded spaces, see Theorem 13. In this vein, that is by substituting arbitrary L-embedded spaces for $L^1(\mu)$, we recover also some results of Godefroy, Kalton, Li [10] in §5. Finally, in §6 it is proved that addition is τ_μ -continuous in preduals of von Neumann algebras.

§2 Notation, Background:

The results are stated for complex scalars. The dual of a Banach space X is denoted by X' . B_X denotes the unit ball of X . Subspace of a Banach space means norm-closed subspace, bounded always means norm-bounded. As usual, we consider a Banach space as a subspace of its bidual omitting the canonical embedding. $[x_n]$ denotes the closed linear span of a (finite or infinite) sequence (x_n) .

Basic properties and definitions which are not explained here can be found in [4] or in [20]-[21] for Banach spaces and in [22, 28] for C^* -algebras. The standard reference for M- and L-embedded spaces is the monograph [13].

Let (x_n) be a sequence of nonzero elements in a Banach space X .

We say that (x_n) *spans l^1 isomorphically* (or *r -isomorphically* to be more precise) - $(x_n)_{n \in \mathbb{N}} \overset{r}{\sim} l^1$ or just $x_n \overset{r}{\sim} l^1$ in symbols - if there exists $r > 0$ (trivially $r \leq 1$) such that $r(\sum_{n=1}^{\infty} |\alpha_n|) \leq \|\sum_{n=1}^{\infty} \alpha_n \frac{x_n}{\|x_n\|}\| \leq \sum_{n=1}^{\infty} |\alpha_n|$ for all scalars α_n (the second inequality being trivial).

We say that (x_n) *spans l^1 almost isometrically* - $x_n \overset{\text{alm}}{\sim} l^1$ in symbols - if there is a sequence (δ_m) in $[0, 1[$ tending to 0 such that $(x_n)_{n \geq m} \overset{1-\delta_m}{\sim} l^1$ for all $m \in \mathbb{N}$. Recall that the Banach-Mazur distance of two Banach spaces X and Y is defined by $\text{dist}(X, Y) = \inf \|T\| \|T^{-1}\|$ where the infimum extends over all surjective isomorphisms $T : X \rightarrow Y$. To avoid confusion, notice that $\text{dist}(l^1, [x_n]) = 1$ is not the same as $x_n \overset{\text{alm}}{\sim} l^1$.

Finally, a sequence (x_n) is said to *span l^1 asymptotically isometrically* or just to *span l^1 asymptotically* - $x_n \overset{\text{asy}}{\sim} l^1$ in symbols - if there is a sequence (δ_n) in $[0, 1[$ tending to 0 such that $\sum_{n=1}^{\infty} (1 - \delta_n) |\alpha_n| \leq \|\sum_{n=1}^{\infty} \alpha_n \frac{x_n}{\|x_n\|}\|$ for all scalars α_n .

Note that the present definitions of almost and asymptotically isometric differ slightly from those in [6], [23] by the term $x_n/\|x_n\|$ but that, of course, for normalized sequences the definitions are the same. Note also the technical detail that because of this term one might have $\|x_n\| \rightarrow 0$ for a sequence spanning l^1 isomorphically whereas sequences that are equivalent to the canonical l^1 -basis ([4, p. 43]) are uniformly bounded away from 0. We say that a Banach space is isomorphic (respectively almost isometric respectively asymptotically isometric) to l^1 if it has a basis with the corresponding property.

Let Y be a subspace of a Banach space X and P be a projection on X . P is called an *L-projection* provided $\|x\| = \|Px\| + \|(\text{id}_X - P)x\|$ for all $x \in X$. A subspace $Y \subset X$ is called an *M-ideal in X* if its annihilator Y^\perp in X' is the range of an L-projection on X' . Y is called an *L-summand in X* if it is the range of an L-projection on X . In the special case in which $X = Y''$ and in which Y is an M-summand (respectively an L-summand) in Y'' we say that Y is *M-embedded* (respectively *L-embedded*). As examples we only mention that preduals of von Neumann algebras, in particular l^1 and L^1 -spaces, furthermore the Hardy space H_0^1 and the dual of the disc algebra are L-embedded. The sequence space c_0 , the space of compact operators on a Hilbert space, and the quotient C/A of the continuous functions on the unit circle by the disc algebra A are examples among M-embedded spaces. It is not difficult but important to see that if there is an L-projection P on a Banach space X then each contractive projection on X which has the same kernel as P coincides with P , see [13, Prop. I.1.2]. It follows that if X is M-embedded then the canonical inclusion of X' in X''' is an L-summand in X''' that is X' is L-embedded; the converse is false [13, III.1.3]; in fact, for an L-embedded space being the dual of an M-embedded space can be quite a restrictive condition: For instance, while the dual of any C^* -algebra is L-embedded only those C^* -algebras are M-embedded which are isometrically $*$ -isomorphic to the algebra of compact operators or to a c_0 -sum of such algebras [13, III.2.9]. Throughout this note, if X denotes an L-embedded Banach space (which is not always the case) we will write X_s for the complement of (the canonical embedding of) X in X'' that is $X'' = X \oplus_1 X_s$. In this case P will denote the L-projection from X'' onto X .

We recall Godefroy's fundamental result [9], [13, IV.2.2] that L-embedded Banach spaces are w -sequentially complete. This will be used mostly without reference; together with Rosenthal's l^1 -theorem a typical application is that each bounded sequence in an L-embedded space contains a subsequence which either spans l^1 or converges weakly. There is a useful criterion for L-embeddedness of subspaces of L-embedded spaces due to Li ([19] or [13, Th. IV.1.2]) which we state for easy reference:

Lemma 1 (Li) *For an L-embedded Banach space X (with L-decomposition $X'' = X \oplus_1 X_s$ and L-projection P on X'' with range X) and a closed subspace Y of X the following assertions are equivalent.*

- (i) Y is L-embedded.
- (ii) $Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap X_s)$.
- (iii) $P\overline{BY}^{w*} = B_Y$.
- (iv) $PY^{\perp\perp} = Y$.

In particular if Y is L-embedded and if one identifies $Y'' = Y \oplus_1 Y_s$ and $Y^{\perp\perp} \subset X''$ then $Y_s = Y^{\perp\perp} \cap X_s$.

Let us finally cite some technical results from [23] which will be used in the sequel.

It is routine to show that sequences spanning l^1 asymptotically are stable by adding norm-null sequences [24, Lem. 4], to be more precise, let $(x_n), (y_n)$ be two sequences in a Banach space X such that (x_n) spans l^1 asymptotically, $\inf \|x_n\| > 0$, $\|y_n\| \rightarrow 0$ and $x_n + y_n \neq 0$. Then $(x_n + y_n)$ spans l^1 asymptotically, too.

Although it has been proved in [6] that there are almost isometric l^1 -copies which do not contain asymptotic ones, both notions "coincide up to subsequences" in L-embedded Banach spaces, more precisely, in L-embedded Banach spaces each sequence spanning l^1 almost isometrically contains a subsequence spanning l^1 asymptotically [23, Cor. 3].

The following lemma is fundamental for the rest of the paper. It is an immediate consequence of [23] and says that within L-embedded spaces, sequences spanning l^1 almost or asymptotically isometrically behave like the standard basis of l^1 as to their w^* -accumulation points. Mostly it will be used with M being a countable set of normalized elements that span l^1 asymptotically.

Lemma 2 *Let X be L-embedded (with L-decomposition $X'' = X \oplus_1 X_s$), let $M \subset X$ be a subset of the unit sphere of X . Then the following assertions are equivalent.*

- (i) M contains a sequence spanning l^1 asymptotically.
- (ii) M admits a w^* -accumulation point $x_s \in X_s$ of norm one.

In fact, each w^* -accumulation point of a sequence spanning l^1 asymptotically lies in X_s .

Proof: (i) \Rightarrow (ii) and the second statement: see last paragraph of the proof of [23, Lem. 1]; (ii) \Rightarrow (i): [23, Th. 2] ■

§3 Abstract measure topology

To prepare the definition of an abstract measure topology we recall some facts about sequential spaces (see for example [8, 1.6-1.7] or [18], cf. also [17]) because the topology will be defined by determining the class of its convergent sequences.

A topological space is called a sequential space if closedness and sequential closedness coincide. A topological space is called a Fréchet space if closure and sequential closure coincide. Clearly first countable spaces are Fréchet spaces and Fréchet spaces are sequential spaces. Recall that a topological space is a T_1 -space if every one-point set is closed; this happens if each convergent sequence has a unique limit.

A Li^* -space¹ is a triple (X, \mathcal{C}, li^*) where X is a set, $\mathcal{C} \subset X^{\mathbb{N}}$ a class of sequences of X (called the convergence class) and $li^* : \mathcal{C} \rightarrow X$ a map (called limit operator) satisfying the following conditions (L1)-(L3). We write li^*x_n instead of $li^*((x_n)_{n \in \mathbb{N}})$; the elements of \mathcal{C} are called \mathcal{C} -convergent sequences. Let (x_n) be any sequence in X , let $x \in X$.

- (L1) If $x_n = x$ for all $n \in \mathbb{N}$ then $(x_n) \in \mathcal{C}$ and $li^*x_n = x$.
- (L2) If $(x_n) \in \mathcal{C}$ with $li^*x_n = x$ then $(x_{n_k}) \in \mathcal{C}$ and $li^*x_{n_k} = x$ for each subsequence (x_{n_k}) of (x_n) .
- (L3) If a sequence (x_n) is such that there is $x \in X$ and any subsequence (x_{n_k}) contains a further subsequence $(x_{n_{k_m}})$ such that $(x_{n_{k_m}}) \in \mathcal{C}$ and $li^*x_{n_{k_m}} = x$ then $(x_n) \in \mathcal{C}$ and $li^*x_n = x$.

On a Li^* -space (X, \mathcal{C}, li^*) one defines a topology τ_{li^*} - called the sequential topology induced by li^* - by taking as the family of closed sets all li^* -sequentially closed sets; here we call a set A li^* -sequentially closed if $li^*x_n \in A$ for all \mathcal{C} -convergent sequences (x_n) contained in A . It is elementary to verify that in this way one indeed obtains a topology and that the τ_{li^*} -convergent sequences are exactly the \mathcal{C} -convergent sequences, that is for every sequence (x_n) in X one has $li^*x_n = x$ if and only if $x_n \xrightarrow{\tau_{li^*}} x$.

An \mathcal{S}^* -space is a Li^* -space (X, \mathcal{C}, li^*) satisfying additionally

- (L4) If $(x_m^{(n)})_m \in \mathcal{C}$ for all $n \in \mathbb{N}$ and $(x_n) \in \mathcal{C}$ such that $li^*x_m^{(n)} = x_n$ for all $n \in \mathbb{N}$ and $li^*x_n = x$ then there exist two sequences $(n_k), (m_k)$ such that $li^*x_{m_k}^{n_k} = x$.

Endowed with τ_{li^*} , an \mathcal{S}^* -space becomes a Fréchet space [8, 1.7.18, 19].

As already mentioned in the introduction the following definition is patterned after a characterization of bounded measure-null sequences in the preduals of finite von Neumann algebras [24, Th. 1].

Definition 3 Let X be a Banach space. A system τ_μ of subsets of X is called an abstract measure topology if it satisfies the following four conditions.

1. (X, τ_μ) is a sequential space in which every convergent sequence has a unique limit.
2. τ_μ is weaker than the norm topology.
3. τ_μ is translation invariant for sequences more precisely, $x_n \xrightarrow{\tau_\mu} x$ if and only if $x_n - x \xrightarrow{\tau_\mu} 0$ for any sequence (x_n) in X .
4. Each bounded sequence in X that spans l^1 asymptotically τ_μ -converges to 0, and each sequence in X that τ_μ -converges to 0 is bounded and contains a subsequence which spans l^1 asymptotically or tends to 0 in norm.

The following result is quite easy to prove. Nevertheless, because of its importance, we call it a theorem.

¹To avoid confusion with the letter L like in L-embedded, L-projection, L-structure we prefer the notation Li^* instead of \mathcal{L}^* as in [8, 18].

Theorem 4 *Every L-embedded Banach space admits an abstract measure topology.*

Proof: Let X be an L-embedded Banach space with L-decomposition $X'' = X \oplus_1 X_s$. Set

$$\begin{aligned} \mathcal{C}_0 &= \{(x_n) \mid (x_n) \text{ is bounded and every subsequence } (x_{n_k}) \text{ of } (x_n) \text{ contains} \\ &\quad \text{a subsequence } (x_{n_{k_l}}) \text{ such that } x_{n_{k_l}} \overset{\text{asy}}{\sim} l^1 \text{ or } \|x_{n_{k_l}}\| \rightarrow 0\}, \\ \mathcal{C} &= \{(x_n) \mid \text{there exists } x \in X \text{ such that } (x_n - x) \in \mathcal{C}_0\}. \end{aligned}$$

We define a limit operator $li^* : \mathcal{C} \rightarrow X$ by $li^* x_n = x$ where $x \in X$ is such that $(x_n - x) \in \mathcal{C}_0$. To show that (X, \mathcal{C}, li^*) is a Li^* -space the only thing to verify is that li^* is well defined as a map. because then conditions (L1) - (L3) are immediate from the definition of \mathcal{C} .

Suppose that there are $x, y \in X$, $(x_n) \in \mathcal{C}$ such that both $(x_n - x) \in \mathcal{C}_0$ and $(x_n - y) \in \mathcal{C}_0$. If $(x_n - x)$ or $(x_n - y)$ admits a subsequence tending to 0 in norm then $x = y$ because the norm topology is Hausdorff. Otherwise, after passing to an appropriate subsequence, we suppose that both sequences are uniformly bounded away from 0 in norm and that both $x_n - x \overset{\text{asy}}{\sim} l^1$ and $x_n - y \overset{\text{asy}}{\sim} l^1$. Since both sequences are bounded, by Lemma 2 they admit two w^* -accumulation points $x_s, y_s \in X_s$ and there is a net (x_{n_γ}) such that $x_{n_\gamma} - x \xrightarrow{w^*} x_s$ and $x_{n_\gamma} - y \xrightarrow{w^*} y_s$. But this means that $x_{n_\gamma} \xrightarrow{w^*} x + x_s = y + y_s$ whence $x = y$ (and $x_s = y_s$).

We define the abstract measure topology τ_μ as the sequential topology induced by li^* . It is immediate from the definition of \mathcal{C} that τ_μ satisfies the conditions of Definition 3. ■

Lemma 5 shows some elementary properties of τ_μ which in the sequel will be used mostly without reference.

Lemma 5 *If a Banach space X admits an abstract measure topology τ_μ then τ_μ has the following properties.*

- (a) (X, τ_μ) is a T_1 -space.
- (b) The relative topology of τ_μ on a subspace of X is again an abstract measure topology.
- (c) Closedness and sequential closedness coincide for τ_μ . Sequentially continuous maps on X are continuous.
- (d) If X does not contain a copy of l^1 the norm topology is an abstract measure topology and is the only one.
- (e) τ_μ is unique.
- (f) If X is the predual of a finite von Neumann algebra then τ_μ coincides on bounded sets with the usual measure topology; on unbounded sets it does not in general.
- (g) Multiplication by scalars is τ_μ -continuous.
- (h) If X is L-embedded (i.e. $X'' = X \oplus_1 X_s$) and if a net (x_γ) in X w^* -converges to $x_s \in X_s$ such that $\|x_\gamma\| \leq \|x_s\|$ then $x_\gamma \xrightarrow{\tau_\mu} 0$.
- (i) If X is L-embedded then for any $x'' \in \overline{B_{X_s}}^{w^*}$ there exists a net (x_γ) in B_X such that both $x_\gamma \xrightarrow{\tau_\mu} 0$ and $x_\gamma \xrightarrow{w^*} x''$.

Proof: (a) - (d) are clear from the definition.

(e) The topology of sequential spaces is determined by its convergent sequences. Thus the abstract measure topology is unique because the conditions of Definition 3 determine all convergent sequences.

(f) The assertion concerning bounded sets follows from [24, Th. 1]. For the assertion concerning unbounded sets we consider the usual measure (=pointwise) topology and the w^* -topology on l^1 : These two and τ_μ coincide on bounded sets. But the unbounded sequence (ne_n) converges in the usual measure topology while it does not with respect to τ_μ . (Here (e_n) denotes the standard basis of l^1 .)

(g) Let $\lambda_n \rightarrow \lambda$ in \mathbb{C} , and $x_n \xrightarrow{\tau_\mu} x$ in X . If $\lambda_n \rightarrow 0$ or $\|x_n - x\| \rightarrow 0$ then $\lambda_n x_n \rightarrow \lambda x$ with respect to the norm-topology and thus also with respect to τ_μ . If $\inf |\lambda_n| > 0$, $x_n - x \overset{\text{asy}}{\sim} l^1$, and $\inf \|x_n - x\| > 0$ then $\lambda_n x_n - \lambda_n x \overset{\text{asy}}{\sim} l^1$ whence $\lambda_n x_n - \lambda x \overset{\text{asy}}{\sim} l^1$ because $\|\lambda_n x - \lambda x\| \rightarrow 0$ and because sequences spanning l^1 asymptotically are stable by adding norm-null sequences. Hence $\lambda_n x_n \xrightarrow{\tau_\mu} \lambda x$. Up to the usual reasoning on subsequences of subsequences this proves that $\lambda_n \rightarrow \lambda$ and $x_n \xrightarrow{\tau_\mu} x$ imply $\lambda_n x_n \xrightarrow{\tau_\mu} \lambda x$.

(h) We assume to the contrary that (x_γ) does not τ_μ -converge to 0. Then there is a τ_μ -neighborhood \mathcal{O} of 0 and a subnet $(x_{\gamma'})$ which does not meet \mathcal{O} and which still w^* -converges to x_s . But by Lemma 2 there is a

sequence $(x_{\gamma'_n})$ that spans l^1 almost isometrically i.e. $x_{\gamma'_n} \xrightarrow{\tau_\mu} 0$ hence $(x_{\gamma'_n})$ does meet \mathcal{O} .

(i) Let \mathcal{U} be a w^* -neighbourhood of x'' . Then there exists a $x_s \in B_{X_s} \cap \mathcal{U}$ and \mathcal{U} is also a w^* -neighbourhood of x_s . Let $(x_\gamma) \subset B_X$ be a net with w^* -limit x_s and such that $\|x_\gamma\| = \|x_s\|$. Let \mathcal{V} be a τ_μ -neighbourhood of 0. Then $x_\gamma \xrightarrow{\tau_\mu} 0$ by (h) and there is γ_0 such that $x_\gamma \in \mathcal{U}$ and $x_\gamma \in \mathcal{V}$ for all $\gamma \succeq \gamma_0$. That is $\mathcal{V} \cap \mathcal{U} \cap B_X \neq \emptyset$ whence the assertion. ■

Remarks:

1. Part (h) of the Lemma above corresponds to [13, IV.3.7], part (i) to [10, Lem. 2.2]. Since in general B_{X_s} is not w^* -closed there are τ_μ -null sequences which differ from the ones described in part (h). It is tempting to suppose that the τ_μ -null nets are exactly those that admit w^* -limits in $\overline{B_{X_s}}^{w^*}$ but at the time of this writing this is not at all clear.
2. There exists a non-Hausdorff Fréchet space in which every sequence has at most one limit [8, 1.6E]. Therefore the question whether τ_μ is Hausdorff is not necessarily trivial.
3. If X, Y are two L-embedded Banach spaces, $Y \subset X$ a subspace of X , then by (b) and (e) one can identify the intrinsic abstract measure topology of Y with the relative topology of the abstract measure topology of X . Therefore, in theorems like Theorem 13 or Proposition 16 it is no longer necessary to consider a surrounding L-embedded Banach space like $L^1(\mu)$ in the corresponding theorems [13, IV.3.10] or [10, Prop. 2.1]. This observation sheds also some light on the remarks after [13, IV.3.5] and after [13, Def. IV.4.2] on the use of "nicely placed" and "L-embedded".
4. It might be useful for other purposes to modify Definition 3. For example one could replace "asymptotically" by "almost isometrically" in Definition 3; by [23, Cor. 3] this would give the same topology for L-embedded spaces. As a less trivial modification one could first restrict Definition 3 to bounded subsets of a Banach space X and then define the abstract measure topology on the whole of X as the inductive limit of the family $\tau_\mu|_{nB_x}$. In this case, the results of this paper would remain valid up to some minor modifications because in all proofs except for Lemma 5 only the restriction of τ_μ to the unit ball is considered. In passing we note that we restrict our attention to bounded sets mainly because the characterization of measure-null sequences in [24] does not work for unbounded sequences, see the second remark after the proof of Theorem 1 in [24].

We end this section with a modest attempt to get closer to the sequential structure of τ_μ . In case the addition is τ_μ -continuous, at least on bounded sets τ_μ gives a Fréchet space. This applies to von Neumann preduals which in §6 below will be shown to have τ_μ -continuous addition.

Lemma 6 *Let X be an L-embedded Banach space. If the addition is τ_μ -continuous then the restriction of τ_μ to a bounded subset of X makes this set a Fréchet space.*

Proof: To avoid trivialities we consider a bounded sequence $x_n \xrightarrow{\text{asy}} l^1$ in X , and a uniformly bounded sequence of sequences $(x_m^{(n)})_{m \in \mathbb{N}}$ such that $x_m^{(n)} - x_n \xrightarrow{\text{asy}} l^1$ for all $n \in \mathbb{N}$. It is enough to show the existence of a sequence (m_n) such that $x_{m_n}^{(n)} \xrightarrow{\tau_\mu} 0$ because this will prove that (X, \mathcal{C}, l^1) of the proof of Theorem 4 satisfies (L4).

We set $y_m^{(n)} = x_m^{(n)} - x_n$. Since all $x_m^{(n)}$ are uniformly bounded there is no loss of generality if we suppose that $\|y_m^{(n)}\| = 1$ for all m and all n . By Lemma 2 each universal net $(y_{m_\gamma}^{(n)})_{\gamma \in \Gamma}$ w^* -converges to a limit $y_s^{(n)} \in X_s$ of norm one. What follows is a straightforward modification of the proof of [23, Th. 2]. Let (δ_n) be a sequence of strictly positive numbers in $]0, 1]$ converging to 0. Set $\eta_1 = \frac{1}{3}\delta_1$ and $\eta_{n+1} = \frac{1}{3} \min(\eta_n, \delta_{n+1})$ for $n \in \mathbb{N}$. By induction over $n \in \mathbb{N}$ one constructs $m_n \in \mathbb{N}$ such that

$$\left(\sum_{i=1}^n (1 - \delta_i) |\alpha_i| \right) + \eta_n \sum_{i=1}^n |\alpha_i| \leq \left\| \sum_{i=1}^n \alpha_i y_{m_i}^{(i)} \right\| \text{ for all } n \in \mathbb{N}, \alpha_i \in \mathbb{C}. \quad (1)$$

The first induction step is settled by setting $m_1 = 1$. For the induction step $n \mapsto n+1$ fix an element $\alpha = (\alpha_i)_{i=1}^{n+1}$ in the unit sphere of l_{n+1}^1 such that $\alpha_{n+1} \neq 0$. The w^* -convergence (along γ) of $(\sum_{i=1}^n \alpha_i y_{m_i}^{(i)}) + \alpha_{n+1} y_{m_\gamma}^{(n+1)}$ to $(\sum_{i=1}^n \alpha_i y_{m_i}^{(i)}) + \alpha_{n+1} y_s^{(n+1)}$ yields

$$\begin{aligned} \liminf_{\gamma} \left\| \left(\sum_{i=1}^n \alpha_i y_{m_i}^{(i)} \right) + \alpha_{n+1} y_{m_\gamma}^{(n+1)} \right\| &\geq \left\| \left(\sum_{i=1}^n \alpha_i y_{m_i}^{(i)} \right) + \alpha_{n+1} y_s^{(n+1)} \right\| \\ &= \left\| \sum_{i=1}^n \alpha_i y_{m_i}^{(i)} \right\| + |\alpha_{n+1}| \end{aligned}$$

$$\stackrel{(1)}{\geq} \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + \min(\eta_n, \delta_{n+1})$$

whence

$$\left\| \left(\sum_{i=1}^n \alpha_i y_{m_i}^{(i)} \right) + \alpha_{n+1} y_m^{(n+1)} \right\| \geq \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + 3\eta_{n+1}$$

for infinitely many m . This extends to a finite η_{n+1} -net $(\alpha^l)_{l=1}^{L_{n+1}}$ in the unit sphere of l_{n+1}^1 with $2\eta_{n+1}$ instead of $3\eta_{n+1}$, to all α in the unit sphere of l_{n+1}^1 with η_{n+1} instead of $2\eta_{n+1}$, and finally to all $\alpha \in l_{n+1}^1$. The details are the same as in the proof of [23, Th. 2].

This ends the induction. By (1) we have $y_{m_n}^{(n)} \xrightarrow{\tau_\mu} 0$. Hence $x_{m_n}^{(n)} = y_{m_n}^{(n)} + x_n \xrightarrow{\tau_\mu} 0$ because addition is supposed to be τ_μ -continuous. ■

§4 Section IV.3 of [13] (partly) revisited

Proposition 7 generalizes some well known facts, see e.g. [7, Th. IV.8.12] for (a) and [13, p. 202] for (b).

Proposition 7 *Let X be an L -embedded Banach space with its abstract measure topology τ_μ . Then the following statements hold.*

- (a) *A sequence converges in norm if (and only if) it converges both weakly and with respect to τ_μ , and all limits coincide.*
- (b) *A norm closed subspace $Y \subset X$ is reflexive if and only if τ_μ and the norm topology coincide on the unit ball of Y .*

Proof. (a) The statement is almost immediate from the definition of τ_μ : First remark that a sequence which τ_μ -converges to 0 contains a subsequence which either converges to 0 in norm or is uniformly bounded away from 0 in norm and spans l^1 (asymptotically); but the latter case is excluded if the sequence also converges weakly (to whatever limit) because l^1 -bases do not converge weakly. Now let (x_n) be a sequence in X , let $x, y \in X$ be such that both $x_n \xrightarrow{\tau_\mu} x$ and $x_n \xrightarrow{w} y$. Then by what has just been remarked, for each subsequence (x_{n_k}) there is a subsequence $(x_{n_{k_l}})$ such that $x_{n_{k_l}} - x \rightarrow 0$ in norm for any subsequence (x_{n_k}) whence $x = y$ and the assertion follows.

(b) By $\tau_{\|\cdot\|}$ we denote the norm topology on X . Let Y be reflexive. To show that τ_μ and $\tau_{\|\cdot\|}$ coincide on the unit ball B_Y of Y it is enough to show that each subsequence of a τ_μ -convergent sequence in B_Y admits a subsequence which converges in norm to the same limit. But if Y is reflexive then each bounded sequence contains a weakly convergent subsequence which if the sequence is also τ_μ -convergent converges in norm by (a). Thus τ_μ and $\tau_{\|\cdot\|}$ coincide on B_Y .

Conversely suppose that τ_μ and $\tau_{\|\cdot\|}$ coincide on B_Y . In order to prove that Y is reflexive it is enough to prove that Y does not contain isomorphic copies of l^1 because by Rosenthal's theorem [4, Ch. XI] in the absence of l^1 each bounded sequence contains a weak Cauchy subsequence which then converges weakly by the weak sequential completeness of X . But if Y contained an isomorphic copy of l^1 then by James' distortion theorem it would contain also an almost isometric copy of l^1 . By [23, Cor. 3] Y would contain an asymptotic copy of l^1 . Its canonical normalized basis would τ_μ -converge to 0; finally, since τ_μ and $\tau_{\|\cdot\|}$ coincide on B_Y this basis would even converge in norm to 0, a contradiction which proves that Y is reflexive. ■

Lemma 8 is the technical key for the other results of this section. It corresponds to [13, IV.3.1].

Lemma 8 *Let X be an L -embedded Banach space with L -projection P from X'' onto X . Then for every net $(x_\gamma)_{\gamma \in \Gamma}$ in X w^* -converging to $x'' \in X'' \setminus X$ there is a bounded sequence (y_n) in $\text{co}\{x_\gamma \mid \gamma \in \Gamma\}$ such that the sequence $(y_n - Px'')$ spans l^1 asymptotically isometrically.*

Proof. Set $x = Px''$, $x_s = x'' - Px''$. Choose a net $(z_\gamma)_{\gamma \in \Gamma}$ in X such that $z_\gamma \xrightarrow{w^*} x_s$ and $\|z_\gamma\| = \|x_s\|$. We assume without loss of generality that both nets (x_γ) , (z_γ) are indexed by the same directed set Γ . Then $x_\gamma - z_\gamma \xrightarrow{w} x$.

The idea of the proof is that on one hand by the theorem of Hahn-Banach the net $(x_\gamma - z_\gamma)$ admits convex combinations which converge to x in norm and that on the other hand by a slight modification of Godefroy's construction the corresponding convex combinations of the x_γ can be chosen so to span l^1 asymptotically.

Here are the details.

Since $x'' \notin X$ we have $x_s \neq 0$ and thus without loss of generality we suppose $\|x_s\| = \|z_\gamma\| = 1$. Let (δ_n) be a sequence of numbers in $]0, 1[$ convergent to 0. Set $\eta_1 = \frac{1}{4}\delta_1$ and $\eta_{n+1} = \frac{1}{4}\min(\eta_n, \delta_{n+1})$ for $n \in \mathbb{N}$. By induction over $n \in \mathbb{N}$ we will construct finite sets $A_n \subset \mathbb{N}$, finite sequences $(\lambda_k)_{k \in A_n}$ in $[0, 1]$ and $(\gamma_k)_{k \in A_n}$ in Γ such that

$$\sum_{k \in A_n} \lambda_k = 1, \quad G_i \cap G_n = \emptyset, \quad G_n \subset \Gamma_n \quad \forall i < n \quad (2)$$

$$\|(y_n - x) - \sum_{k \in A_n} \lambda_k z_{\gamma_k}\| < \eta_n. \quad (3)$$

$$\sum_{i=1}^n |\alpha_i|(1 + \eta_i) \geq \left\| \sum_{i=1}^n \alpha_i (y_i - x) \right\| \geq \left(\sum_{i=1}^n (1 - \delta_i) |\alpha_i| \right) + \eta_n \sum_{i=1}^n |\alpha_i| \quad \forall \alpha_i \in \mathbb{C} \quad (4)$$

where

$$y_n = \sum_{k \in A_n} \lambda_k x_{\gamma_k}, \quad G_n = \{\gamma_k \mid k \in A_n\}, \quad \Gamma_n = \Gamma \setminus \bigcup_{i=1}^{n-1} G_i \quad (= \Gamma \text{ if } n = 1).$$

For the first induction step $n = 1$ we choose $x' \in B_{X'}$ such that $1 = \|x_s\| \geq \operatorname{Re} x_s(x') > 1 - \delta_1 + 2\eta_1$. The w^* -convergence of (z_γ) to x_s yields $\beta_1 \in \Gamma$ such that

$$\operatorname{Re} x'(z_\gamma) > 1 - \delta_1 + 2\eta_1 \quad \forall \gamma \succeq \beta_1. \quad (5)$$

The net $((x_\gamma - x) - z_\gamma)_{\gamma \succeq \beta_1}$ w -converges to 0 thus by the theorem of Hahn-Banach we find a convex combination $y_1 = \sum_{k \in A_1} \lambda_k x_{\gamma_k}$ such that

$$\|(y_1 - x) - \sum_{k \in A_1} \lambda_k z_{\gamma_k}\| < \eta_1.$$

Thus (4, $n = 1$) follows from $\|y_1 - x\| \leq \|\sum_{k \in A_1} \lambda_k z_{\gamma_k}\| + \eta_1 \leq 1 + \eta_1$ and from

$$\begin{aligned} \|y_1 - x\| &> \left\| \sum_{k \in A_1} \lambda_k z_{\gamma_k} \right\| - \eta_1 \\ &\geq -\eta_1 + \operatorname{Re} \sum_{k \in A_1} \lambda_k x'(z_{\gamma_k}) \\ &> -\eta_1 + \sum_{k \in A_1} \lambda_k (1 - \delta_1 + 2\eta_1) = 1 - \delta_1 + \eta_1. \end{aligned}$$

For the induction step $n \mapsto n + 1$ we suppose $A_i \subset \Gamma$, $(\lambda_k)_{k \in A_i} \subset [0, 1]$, $G_i \subset \Gamma$ to be constructed according to (2) - (4) for $i = 1, \dots, n$.

First we consider the w -convergent net $(x_\gamma - z_\gamma)_{\gamma \in \Gamma_{n+1}}$. By the theorem of Hahn-Banach we can choose a finite set $A_{n+1} \subset \mathbb{N}$, numbers $(\lambda_k)_{k \in A_{n+1}} \subset [0, 1]$ and indices $(\gamma_k)_{k \in A_{n+1}} \subset \Gamma_{n+1}$ such that (2) and (3) hold for $n + 1$. Together with (4, n) this gives the first inequality of (4, $n + 1$).

We fix an element $\alpha = (\alpha_i)$ in the unit sphere of l_{n+1}^1 such that $\alpha_{n+1} \neq 0$ and use the L-decomposition of $X'' = X \oplus_1 X_s$ in order to get

$$\begin{aligned} \left\| \left(\sum_{i=1}^n \alpha_i (y_i - x) \right) + \alpha_{n+1} x_s \right\| &= \left\| \sum_{i=1}^n \alpha_i (y_i - x) \right\| + \|\alpha_{n+1} x_s\| \\ &\stackrel{(4)}{\geq} \left(\sum_{i=1}^n (1 - \delta_i) |\alpha_i| \right) + \eta_n \left(\sum_{i=1}^n |\alpha_i| \right) + |\alpha_{n+1}| \\ &= \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + \eta_n - (\eta_n - \delta_{n+1}) |\alpha_{n+1}| \\ &\geq \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + \min(\eta_n, \delta_{n+1}) \\ &= \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + 4\eta_{n+1} \end{aligned}$$

because $\|\alpha\| = 1$ and $|\alpha_{n+1}| \leq 1$. Thus there is $x' \in B_{X'}$ (depending on α) such that

$$\begin{aligned} \left\| \left(\sum_{i=1}^n \alpha_i (y_i - x) \right) + \alpha_{n+1} x_s \right\| &\geq \operatorname{Re} \left(\left(\sum_{i=1}^n \alpha_i (y_i - x) \right) + \alpha_{n+1} x_s \right) (x') \\ &> \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + 3\eta_{n+1}. \end{aligned}$$

Then the w^* -convergence (along $\gamma \in \Gamma_{n+1}$) of $\left(\left(\sum_{i=1}^n \alpha_i (y_i - x) \right) + \alpha_{n+1} z_\gamma \right)$ to $\left(\sum_{i=1}^n \alpha_i (y_i - x) \right) + \alpha_{n+1} x_s$ yields $\beta \in \Gamma_{n+1}$ (depending on α and x') such that

$$\operatorname{Re} \left(x' \left(\left(\sum_{i=1}^n \alpha_i (y_i - x) \right) + \alpha_{n+1} z_\gamma \right) \right) > \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + 3\eta_{n+1} \quad \forall \gamma \succeq \beta.$$

Choose a finite η_{n+1} -net $(\alpha^l)_{l=1}^{L_{n+1}}$ in the unit sphere of l_{n+1}^1 in the sense that for each α in the unit sphere of l_{n+1}^1 there is $l \leq L_{n+1}$ such that $\|\alpha - \alpha^l\| = \sum_{i=1}^{n+1} |\alpha_i - \alpha_i^l| < \eta_{n+1}$. Then we may repeat the reasoning above finitely many times for $l = 1, \dots, L_{n+1}$ in order to get $\beta_{n+1} \in \Gamma_{n+1}$ and $x'_l \in B_{X'}$ for $l \leq L_{n+1}$ such that

$$\operatorname{Re} \left(x'_l \left(\left(\sum_{i=1}^n \alpha_i^l (y_i - x) \right) + \alpha_{n+1}^l z_\gamma \right) \right) > \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i^l| \right) + 3\eta_{n+1} \quad \forall l \leq L_{n+1}, \quad \gamma \succeq \beta_{n+1}. \quad (6)$$

For each $l \leq L_{n+1}$ we get that

$$\left\| \sum_{i=1}^{n+1} \alpha_i^l (y_i - x) \right\| \stackrel{(3)}{\geq} -\eta_{n+1} + \left\| \left(\sum_{i=1}^n \alpha_i^l (y_i - x) \right) + \alpha_{n+1}^l \sum_{k \in A_{n+1}} \lambda_k z_{\gamma_k} \right\| \quad (7)$$

$$= -\eta_{n+1} + \left\| \sum_{k \in A_{n+1}} \lambda_k \left(\left(\sum_{i=1}^n \alpha_i^l (y_i - x) \right) + \alpha_{n+1}^l z_{\gamma_k} \right) \right\| \quad (8)$$

$$\geq -\eta_{n+1} + \sum_{k \in A_{n+1}} \lambda_k \operatorname{Re} x'_l \left(\left(\sum_{i=1}^n \alpha_i^l (y_i - x) \right) + \alpha_{n+1}^l z_{\gamma_k} \right) \quad (9)$$

$$\stackrel{(6)}{\geq} \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i^l| \right) + 2\eta_{n+1} \quad (10)$$

For an arbitrary α in the unit sphere of l_{n+1}^1 choose $l \leq L_{n+1}$ such that $\|\alpha - \alpha^l\| < \eta_{n+1}$. Then

$$\begin{aligned} \left\| \sum_{i=1}^{n+1} \alpha_i (y_i - x) \right\| &\geq \left\| \sum_{i=1}^{n+1} \alpha_i^l (y_i - x) \right\| - \left\| \sum_{i=1}^{n+1} (\alpha_i - \alpha_i^l) (y_i - x) \right\| \\ &\geq \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + 2\eta_{n+1} - \|\alpha - \alpha^l\| \\ &\geq \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + \eta_{n+1} \\ &= \left(\sum_{i=1}^{n+1} (1 - \delta_i) |\alpha_i| \right) + \eta_{n+1} \sum_{i=1}^{n+1} |\alpha_i|. \end{aligned}$$

This extends to all scalars $\alpha_i \in \mathbb{C}$ and thus ends the induction. The sequence (y_n) is bounded because of (3). By (4) the sequence $(y_n - x)$ is easily seen to span l^1 asymptotically. This ends the proof. ■

Remark: The proof yields not only $y \in \operatorname{co} \{x_\gamma \mid \gamma \in \Gamma\}$ but separated blocks $y_n = \sum_{k \in A_n} \lambda_k x_{\gamma_k}$ where the sets $\{x_{\gamma_k} \mid k \in A_n\}$ are pairwise disjoint. Moreover one can obtain, given a sequence (γ'_n) in Γ , that $x_{\gamma_k} \succeq \gamma'_n$ for $k \in A_n$.

In general the unit ball of X is not τ_μ -compact; the Rademacher functions r_n in $L^1([0, 1])$ which are bounded without having a measure convergent subsequence provide a counterexample. [If (r_n) contained a measure convergent subsequence this subsequence would admit a norm convergent subsequence by Proposition 7 (a) because (r_n) spans l^2 isomorphically.] But we have:

Corollary 9 *Every bounded sequence in an L-embedded Banach space admits a sequence of convex combinations which converges with respect to the measure topology.*

Proof: Let (x_n) be a bounded sequence in an L-embedded space X and let (x_{n_γ}) be a universal net that w^* -converges to $x'' \in X''$ by the w^* -compactness of $B_{X''}$. If $x'' \in X$ then this net w -converges, admits norm-convergent convex combinations and we are done in this case. Otherwise x'' lies in $X'' \setminus X$ and one applies Lemma 8 to get a sequence (y_n) in $\text{co}\{x_n \mid n \in \mathbb{N}\}$ which τ_μ -converges to Px'' . ■

Corollary 9 corresponds to [13, p. 202]. In this context there is a natural

Question: Does Komlos' theorem hold accordingly? More precisely, given a bounded sequence in an L-embedded space, does it admit a subsequence whose Cesaro (=arithmetic) means converge with respect to the measure topology?

Note that by (a) of Proposition 7 Komlos' theorem implies the weak Banach-Saks property (which by definition claims that a w -convergent sequence admits a subsequence whose Cesaro means converge in norm, see for example [1], [4, p. 112, 121], [2]). By Rosenthal's l^1 -theorem the weak Banach-Saks property is also half a converse to Komlos' theorem, that is by Rosenthal's l^1 -theorem a bounded sequence in an L-embedded space admits a subsequence which is either equivalent to the standard basis of l^1 or converges weakly; but if one supposes the weak Banach-Saks property to hold then in the second case of a w -convergent sequence there are Cesaro means that converge in norm whence with respect to the measure topology.

There is another related

Question: Does the Kadec-Pelczyński subsequence decomposition (sometimes also called the Kadec-Pelczyński splitting lemma) hold accordingly? This lemma says that a bounded sequence (f_n) in $L^1([0, 1])$ admits a subsequence (f_{n_k}) which can be decomposed in the following sense: there are two bounded sequences (g_k) , (h_k) in $L^1([0, 1])$ such that $f_{n_k} = g_k + h_k$, the g_k are pairwise disjoint, and (h_k) converges weakly. In a recent preprint Randrianantoanina [26] showed the Kadec-Pelczyński subsequence decomposition for preduals of von Neumann algebras. Since the latter are known to have the weak Banach-Saks property [2], Komlos' theorem follows almost immediately from Randrianantoanina's result for von Neumann preduals, see Proposition 24 below.

The following theorem generalizes a theorem of Buhvalov-Lozanovskii ([3], [13, IV.3.4]). As in [13, IV.3.4] the implication (ii) \Rightarrow (i) holds also for unbounded C .

Theorem 10 *Let X be an L-embedded Banach space with L-projection P (on X'' with range X) and endowed with its abstract measure topology τ_μ . For a norm closed bounded convex set $C \subset X$ the following two assertions are equivalent.*

- (i) $P\overline{C}^{w^*} = C$ where w^* refers to the w^* -topology of X'' .
- (ii) C is τ_μ -closed.

Proof: (i) \Rightarrow (ii) Take $c_n \in C$, $x \in X$ with $c_n \xrightarrow{\tau_\mu} x$. It is enough to show that $x \in C$ because closedness and sequential closedness coincide for τ_μ . If the τ_μ -null sequence $(c_n - x)$ contains a norm convergent subsequence then we are done because in the norm topology C is closed. Otherwise an appropriate subsequence $(c_{n_k} - x)$ spans l^1 asymptotically and $\inf \|c_{n_k} - x\| > 0$. By w^* -compactness of \overline{C}^{w^*} there is a net $(c_{n_{k_\gamma}} - x)_{\gamma \in \Gamma}$ on $\mathcal{S} = \{c_{n_k} - x \mid k \in \mathbb{N}\}$ that w^* -converges to $\bar{c} - x \in \overline{C}^{w^*} - x$. By Lemma 2 we have that $\bar{c} - x \in X_s$ because C is bounded. Thus $P(\bar{c} - x) = 0$ and $x = Px = P\bar{c} \in C$ by hypothesis.

(ii) \Rightarrow (i) This implication is essentially Lemma 8: If $(c_\gamma)_{\gamma \in \Gamma}$ is a w^* -convergent net in C with limit \bar{c} it is enough to prove that $P\bar{c} \in C$ because the inclusion $C \subset P\overline{C}^{w^*}$ is trivial. If $\bar{c} \in X$ then there is a sequence of convex combinations of the c_γ converging to \bar{c} in norm whence $\bar{c} \in C$. Otherwise, if $\bar{c} \in X'' \setminus X$, by Lemma 8 there is a sequence $(d_n) \in \text{co}\{c_\gamma \mid \gamma \in \Gamma\} \subset C$ which τ_μ -converges to $P\bar{c}$ hence $P\bar{c} \in C$ because C is τ_μ -closed. ■

Corollary 11 corresponds to [13, IV.3.5]) for the case $X = L^1(\Omega, \Sigma, \mu)$.

Corollary 11 *Let X be an L-embedded Banach space endowed with its abstract measure topology τ_μ . Then a norm closed subspace $Y \subset X$ is L-embedded if and only if its unit ball B_Y is τ_μ -closed.*

The proof is immediate from Theorem 10 with $C = B_Y$ and from Li's criterion Lemma 1. ■

Let X be a Banach space admitting an abstract measure topology τ_μ . Then we define

$$X^\# = \{x' \in X' \mid x'|_{B_X} \text{ is } \tau_\mu\text{-continuous}\}.$$

Remarks:

1. $X^\#$ is a closed subspace of X' . (The proof is left to the reader.)
2. If X is a subspace of an L-embedded Banach space then one has $X^\# = X'$ if and only if X does not contain copies of l^1 . For, in the absence of l^1 , τ_μ coincides with the norm topology hence $X^\# = X'$. Conversely, if X contains a copy of l^1 then by James' distortion theorem it contains also an almost isometric copy U of l^1 spanned by a normalized basis (u_n) . Since X is contained in an L-embedded space, by [23, Cor. 3] we (may pass to an appropriate subsequence and) suppose that $u_n \xrightarrow{\tau_\mu} 0$. Let $x' \in X'$ be a Hahn-Banach extension of the functional on U which corresponds to $1 \in l^\infty$. Then x' is not τ_μ -continuous on B_X since $u_n \xrightarrow{\tau_\mu} 0$ but $x'(u_n) \rightarrow 1$. (Compare also with [13, Rem. (b) p. 186].) ■

Proposition 12 *Let X be an L-embedded Banach space (with L-decomposition $X'' = X \oplus_1 X_s$) endowed with its abstract measure topology τ_μ . Then*

$$X^\# = (X_s)_\perp \ (\subset X').$$

Proof: "⊂" Take $x' \in X^\#$ and $x_s \in X_s$. To prove the inclusion we show that $x_s(x') = 0$.

Let (x_γ) be a net that w^* -converges to x_s with $\|x_\gamma\| = \|x_s\|$. But then, by Lemma 2 there is a sequence (x_{γ_n}) that spans l^1 asymptotically. Hence $x_{\gamma_n} \xrightarrow{\tau_\mu} 0$ and $x'(x_{\gamma_n}) \rightarrow 0$ since x' is τ_μ -continuous on bounded sets. This proves that $x''(x') = 0$. (In passing we note that $x_\gamma \xrightarrow{\tau_\mu} 0$ by (h) of Lemma 5 but that one can not infer from this that $x''(x') = 0$ because it is not clear whether a τ_μ -convergent net has a unique limit.)

"⊃" Assume that there is $x' \in (X_s)_\perp$ that is not τ_μ -continuous on B_X . Then by the definition of τ_μ there are $\varepsilon > 0$ and a sequence (x_n) in B_X such that $x_n \xrightarrow{\tau_\mu} 0$ but $|x'(x_n)| > \varepsilon$ for all $n \in \mathbb{N}$. Still by definition of τ_μ and because x' is norm-continuous we suppose that (x_n) spans l^1 almost isometrically. By w^* -compactness of $B_{X''}$ there exists a w^* -accumulation point $x'' \in B_{X''}$ of $\{x_n \mid n \in \mathbb{N}\}$. Let (x_{n_γ}) be a net w^* -converging to x'' . By Lemma 2, $x'' \in X_s$. Thus $x'(x_{n_\gamma}) \rightarrow x''(x') = 0$ by hypothesis. This contradicts $|x'(x_n)| > \varepsilon$ and proves that x' is τ_μ -continuous on B_X . ■

Theorem 13 (see also [13, IV.3.10]) was proved in [11] for nicely placed (=L-embedded) subspaces X of $L^1(\Omega, \Sigma, \mu)$, μ finite. For its proof Proposition 12 plays the same rôle as [13, IV.3.9] in the proof of [13, IV.3.10]. Recall that if an L-embedded space admits a predual then this predual need not be M-embedded ([13, p. 102]).

Theorem 13 *Let X be an L-embedded Banach space endowed with its abstract measure topology τ_μ . The following assertions are equivalent.*

- (i) X is (isometrically isomorphic to) the dual of an M-embedded Banach space Z .
- (ii) $X^\#$ separates X .

If (i) and (ii) are satisfied Z is (isometrically isomorphic to) $X^\#$.

Proof: (i)⇒(ii): By [13, III.1.3] the L-projections on X'' and on Z''' with kernel $X_s = Z^\perp \subset X''$ can be identified. Thus $Z = (Z^\perp)_\perp = (X_s)_\perp = X^\#$ by Proposition 12; in particular, $X^\#$ separates X .

(ii)⇒(i): By [13, IV.1.9] it is enough to show that X_s is w^* -closed in X'' ; then an M-embedded predual of X exists and is isometrically isomorphic to $(X_s)_\perp$ whence to $X^\#$ by Proposition 12. To see that X_s is w^* -closed we take an element

$$\bar{x} = x + x_s \in \overline{X_s}^{w^*} = ((X_s)_\perp)^\perp = (X^\#)^\perp$$

with $x \in X$, $x_s \in X_s$. Then $x = \bar{x} - x_s \in (X^\#)^\perp \cap X = (X^\#)_\perp = \{0\}$ where the latter equality comes from the fact that $X^\#$ separates X . Thus $\bar{x} = x_s \in X_s$ which proves that X_s is w^* -closed in X'' . ■

Analogously to the l^1 -case we say that a sequence (x_n) of nonzero elements in a Banach space X spans c_0 almost (respectively asymptotically) isometrically if there exists a sequence (δ_m) in $[0, 1[$ tending to 0 such that $(1 - \delta_m) \max_{m \leq n \leq m'} |\alpha_n| \leq \left\| \sum_{n=m}^{m'} \alpha_n \frac{x_n}{\|x_n\|} \right\| \leq (1 + \delta_m) \max_{m \leq n \leq m'} |\alpha_n|$ for all $m \leq m'$ (respectively such that $\max_{n \leq m} (1 - \delta_n) |\alpha_n| \leq \left\| \sum_{n=1}^m \alpha_n \frac{x_n}{\|x_n\|} \right\| \leq \max_{n \leq m} (1 + \delta_n) |\alpha_n|$ for all $m \in \mathbb{N}$).

It follows from the proof of [6, Th. 2] that the dual of an asymptotic c_0 -copy is an asymptotic l^1 -copy. A similar argument shows that this remains true if "asymptotic" is replaced by "almost isometric". Analogously we get

Lemma 14 *Let $X = [x_n]$ be an almost isometric copy of l^1 spanned by the normalized basis (x_n) . Let $x'_n \in X'$ be the biorthogonal functionals (that is $x'_n(x_k) = \delta_{n,k}$). Then (x'_n) spans c_0 almost isometrically. The same holds with "almost" replaced by "asymptotically".*

Proof: First we deal with the case where $x_n \stackrel{\text{alm}}{\sim} l^1$.

By hypothesis there is a null sequence $(\delta_m) \subset [0, 1[$ such that

$$(1 - \delta_m) \sum_m |\beta_n| \leq \left\| \sum_m \beta_n x_n \right\| \leq \sum_m |\beta_n|$$

for all scalars β_n . Let $m \leq m'$ be arbitrary in \mathbb{N} . Since $\|x_n\| = 1$ we have the first inequality of

$$\max_{m \leq n \leq m'} |\alpha_n| \leq \left\| \sum_{n=m}^{m'} \alpha_n x'_n \right\| \leq (1 - \delta_m)^{-1} \max_{m \leq n \leq m'} |\alpha_n| \quad (11)$$

for all scalars α_n . For the second inequality we take any $x \in B_X$ of the form $x = \sum \beta_n x_n$; then

$$\begin{aligned} \left| \sum_{n=m}^{m'} \alpha_n x'_n(x) \right| &= \left| \sum_{n=m}^{m'} \alpha_n \beta_n \right| \leq \max_{m \leq n \leq m'} |\alpha_n| \sum_{n=m}^{m'} |\beta_n| \\ &\leq (1 - \delta_m)^{-1} \max_{m \leq n \leq m'} |\alpha_n| \end{aligned}$$

whence the second inequality of (11). It follows from (11) that (x'_n) spans c_0 almost isometrically.

The case in which $x_n \stackrel{\text{asy}}{\sim} l^1$ is proved similarly. ■

Proposition 15 *An L -embedded almost isometric copy of l^1 is the dual of an M -embedded space which is almost isometric to c_0 .*

The statement remains true when "almost" is replaced by "asymptotically".

Proof: Let X be an L -embedded almost isometric l^1 -copy with a normalized canonical basis (x_n) . Let (x'_n) be the biorthogonal functionals of (x_n) that is $x'_n(x_m) = \delta_{n,m}$ for $n, m \in \mathbb{N}$.

Sublemma $x'_n \in X^\#$ for all $n \in \mathbb{N}$.

Proof of the Sublemma: Suppose there is $n_0 \in \mathbb{N}$ such that $x' = x_{n_0} \notin X^\#$. Then there is a sequence $(y_n) \subset B_X$ and there is $\varepsilon > 0$ such that

$$\|y_n\| = 1, \quad y_n \stackrel{\text{asy}}{\sim} l^1, \quad |x'(y_n)| > \varepsilon \text{ for all } n \in \mathbb{N}.$$

By [23, Lem. 1] $Y = [y_n]$ is L -embedded and $Y^{\perp\perp} = Y \oplus_1 Y_s$ with $Y_s = Y^{\perp\perp} \cap X_s$. Let $\delta > 0$ be arbitrary for the moment. Set $X_0 = [x_n]_{n \geq m_0}$ where m_0 is such that $(x_n)_{n \geq m_0} \stackrel{1-\delta}{\sim} l^1$. X_0 is L -embedded by [23, Lem. 1] and we write $X_0^{\perp\perp} = X_0 \oplus_1 (X_0)_s$ with $(X_0)_s = X_0^{\perp\perp} \cap X_s$. Let $y_s \in X''$ be a w^* -accumulation point of $\{y_n \mid n \in \mathbb{N}\}$. By Lemma 2 we have $\|y_s\| = 1$ and $y_s \in Y_s$. We have $y_s \in (X_0)_s$ because X_0 is co-finite-dimensional in X ; furthermore, $|y_s(x')| \geq \varepsilon$. Let $(z_\gamma) \subset B_{X_0}$ be a normalized net w^* -converging to y_s . After passing to an appropriate subnet we suppose that $|(y_s - z_\gamma)(x')| < \varepsilon/2$ for all γ . By Lemma 2 one can extract a sequence (z_{γ_n}) that spans l^1 asymptotically. Then $|x'(z_{\gamma_n})| \geq \varepsilon/2$ for all n . We define an isomorphism $T : X_0 \rightarrow [e_n]_{n \geq m_0}$ by $x_n \mapsto e_n$ where (e_n) is the standard basis of l^1 . Then $\|T\| \leq (1 - \delta)^{-1}$ and $\|T^{-1}\| \leq 1$. There is $m_1 \in \mathbb{N}$ such that $(z_{\gamma_n})_{n \geq m_1} \stackrel{1-\delta}{\sim} l^1$ since $z_{\gamma_n} \stackrel{\text{alm}}{\sim} l^1$. Hence, with the notation $f_n = T z_{\gamma_n}$, we get $(f_n)_{n \geq m_1} \stackrel{1-2\delta}{\sim} l^1$ because

$$\begin{aligned} (1 - 2\delta) \sum_{n=m_1}^{\infty} |\alpha_n| &< (1 - \delta)^2 \sum_{n=m_1}^{\infty} |\alpha_n| \leq (1 - \delta) \sum_{n=m_1}^{\infty} \frac{|\alpha_n|}{\|f_n\|} \\ &\leq \left\| \sum_{n=m_1}^{\infty} \frac{\alpha_n}{\|f_n\|} z_{\gamma_n} \right\| \leq \left\| \sum_{n=m_1}^{\infty} \alpha_n \frac{f_n}{\|f_n\|} \right\|. \end{aligned}$$

Then by [5, Th. B] or [24, L. 10, 6] there is a sequence $(\tilde{f}_k) \subset l^1$ of pairwise disjointly supported elements of l^1 and a subsequence (f_{n_k}) such that

$$\|f_{n_k} - \tilde{f}_k\| < \delta'$$

where $\delta' \rightarrow 0$ as $\delta \rightarrow 0$. Set $\tilde{z}_k = T^{-1}\tilde{f}_k$. Then

$$\|\tilde{z}_k - z_{\gamma_{n_k}}\| = \|T^{-1}\tilde{f}_k - T^{-1}f_{n_k}\| \leq \|\tilde{f}_k - f_{n_k}\| < \delta'.$$

Now we choose $\delta > 0$ small enough in order to have $\delta'\|x'\| < \varepsilon/4$. Hence

$$|x'(\tilde{z}_k)| \geq \frac{\varepsilon}{2} - |x'(\tilde{z}_k - z_{\gamma_{n_k}})| \geq \frac{\varepsilon}{2} - \delta'\|x'\| \geq \frac{\varepsilon}{4}$$

for all $k \in \mathbb{N}$. On the other hand, for $e' = e'_{n_0} = (T^{-1})'(x')$ we have $e'(e_n) = \delta_{n_0,n}$ that is

$$x'(\tilde{z}_k) = e'(\tilde{f}_k) = 0$$

for all but possibly one $k \in \mathbb{N}$ because the \tilde{f}_k are pairwise disjoint. This contradiction proves the Sublemma. \blacksquare

Since the biorthogonal functionals separate X the Sublemma and Theorem 13 imply that $X^\#$ is M-embedded and $X = (X^\#)'$. The Sublemma states that $[x'_n] \subset X^\#$. In fact one has $[x'_n] = X^\#$. To see this let $z' \in X^\# \setminus [x'_n]$. Since X' is isomorphic to l^∞ there is an infinite set $N' \subset \mathbb{N}$ and there is $\varepsilon > 0$ such that $|z'(x_n)| > \varepsilon$ for all $n \in N'$. But $(x_n)_{n \in N'} \stackrel{\text{alm}}{\sim} l^1$ whence $(x_n)_{n \in N'} \stackrel{\text{asy}}{\sim} l^1$ for an appropriate infinite set $N \subset N'$ by [23, Cor. 3]. This means that $(x_n)_{n \in N}$ τ_μ -converges to 0 hence $z' \notin X^\#$.

We have proved that if X is L-embedded and almost isometric to l^1 then X is the dual of the M-embedded space $X^\#$ and $X^\# = [x'_n]$. To end the proof it just remains to apply Proposition 14.

The arguments are similar for the case in which $x_n \stackrel{\text{asy}}{\sim} l^1$. \blacksquare

§5 On some results of Godefroy, Kalton, Li

First we deal with [10, Prop. 2.1]. The first part of that proposition says that $X^\#$ where X is a nicely placed (=L-embedded) subspace of $L^1(\mu)$ is always M-embedded, not only in the situation of Theorem 13.

Proposition 16 *Let X be an L-embedded Banach space. Then $X^\#$ is M-embedded.*

Proof: With the usual notation $X'' = X \oplus_1 X_s$ and with Proposition 12 we have

$$X^\# = (X_s)_\perp \subset X'.$$

We set

$$Z = \overline{X_s}^{w*} = (X^\#)^\perp \subset X''$$

and

$$Y = X \cap \overline{X_s}^{w*} = (X^\#)_\perp \subset X.$$

By Corollary 11, Y is L-embedded because its unit ball

$$B_Y = B_X \cap \bigcap_{x' \in X^\#} \ker x'$$

is τ_μ -closed. Hence $Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap X_s)$ by Lemma 1. Now the fact $Z = Y \oplus_1 X_s$ and the fact that $X^\#$ is an M-ideal in its bidual Y^\perp can be deduced exactly as in the proof of [10, Prop. 2.1]. \blacksquare

For the proof of Proposition 17 we recall property (m_1^*) . In [16] a separable Banach space Z is defined to have property (m_1^*) if for all $z', z'_n \in Z'$

$$\limsup \|z' + z'_n\| = \|z'\| + \limsup \|z'_n\| \quad (12)$$

whenever $z'_n \xrightarrow{w*} 0$. Analogously a separable Banach space X is defined to have property (m_1) if for all $x, x_n \in X$

$$\limsup \|x + x_n\| = \|x\| + \limsup \|x_n\| \quad (13)$$

whenever $x_n \xrightarrow{w} 0$.

The second part of [10, Prop. 2.1] reads as follows in our context.

Proposition 17 *Let X be a separable L -embedded Banach space. If B_X is τ_μ -sequentially compact then for any $\varepsilon > 0$ there is a subspace X_ε of c_0 such that $\text{dist}(X^\#, X_\varepsilon) < 1 + \varepsilon$.*

Proof: Exactly as in [10] one distinguishes three steps: Firstly one proves that $X^\#$ has property (m_1^*) , secondly one deduces from this property (m_∞^*) and thirdly it remains to apply [16, Th. 3.5]. Only the first step must be modified a bit.

From the proof of Proposition 16 we know that $(X^\#)' = X/Y$ where $Y = X \cap \overline{X_s}^{w^*} = (X^\#)_\perp \subset X$. Let $(u_n) \subset X/Y = (X^\#)'$ be a w^* -null sequence. We denote by $Q : X \rightarrow X/Y$ the quotient map. Let $(x_n) \subset X$ be a bounded sequence such that $Qx_n = u_n$. By hypothesis there is a τ_μ -convergent subsequence - still denoted by (x_n) - such that $x_n - x_0 \overset{\text{asy}}{\sim} l^1$ where $x_0 = \tau_\mu - \lim x_n$ and such that $\lim \|x_0 - x_n\|$ exists. We have $x_0 \in Y$ because for any $x' \in X^\#$ one has

$$x'(x_0) = \lim x'(x_n) = \lim x'(u_n) = 0.$$

We have furthermore that

$$\limsup \|y + x + (x_n - x_0)\| \geq \|y + x\| + \lim \|x_n - x_0\| \quad (14)$$

for all $x \in X$, $y \in Y$. To see this, recall that X is L -embedded, and that by Lemma 2 each universal net $(x_{n_\gamma} - x_0)$ w^* -converges to a limit $x_s \in X_s$ such that $\|x_s\| = \lim_\gamma \|x_{n_\gamma} - x_0\|$ and

$$\lim_\gamma \|y + x + (x_{n_\gamma} - x_0)\| \geq \|y + x + x_s\| = \|y + x\| + \|x_s\|$$

by w^* -continuity of the norm whence (14).

Since $\|Qx\| = \inf_{y \in Y} \|y + x\|$ we deduce from (14) that

$$\limsup \|y + x + (x_n - x_0)\| \geq \|Qx\| + \lim \|u_n\|$$

and

$$\limsup \|Qx + u_n\| \geq \|Qx\| + \lim \|u_n\|$$

which proves that $X^\#$ has property (m_1^*) . The deduction of property (m_∞^*) and the conclusion via [16, Th. 3.5] do not depend on the measure topology and coincide therefore with the arguments in [10]. ■

The following remark gives a characterization of property (m_1^*) in L -embedded Banach spaces. We will not need it in the sequel and state it only because the way we prove it by constructing asymptotic l^1 -sequences fits naturally in the main theme of this paper. Note that the implication (i) \Rightarrow (ii) holds for arbitrary Banach spaces Z , that the implications (i) \Rightarrow (ii) \Leftrightarrow (iii) hold whenever Z is such that its dual admits an abstract measure topology, and that the implication (iii) \Rightarrow (iv) does not need the M -embeddedness of Z . Note furthermore that in Remark 18 the separation assumption on Z could be omitted because the definition of properties (m_1^*) and (m_1) makes sense also for non-separable spaces.

Remark 18

(a) *Let Z be a Banach space such that its dual is L -embedded. Then the following assertions are equivalent:*

- (i) *Z has property (m_1^*) .*
- (ii) *Each w^* -null sequence in Z' admits of a subsequence that converges to 0 in norm or spans l^1 asymptotically.*
- (iii) *Each w^* -null sequence in Z' is τ_μ -null.*

If Z is even separable and M -embedded then the assertions above are equivalent to

- (iv) *$B_{Z'}$ is τ_μ -sequentially compact.*

(b) *An arbitrary Banach space X has property (m_1) if and only if it has the Schur property.*

Proof: (a) We set $X = Z'$. Sketch of

(i) \Rightarrow (ii): The proof resembles the one of [23, Th. 2] the only difference being that (12) replaces the w^* -lower semicontinuity of the norm and the L-embeddedness of Z' . Let $(x_m) \subset X$ be a w^* -null sequence, suppose without loss of generality that $\|x_m\| = 1$. Let (δ_n) be a sequence of strictly positive numbers converging to 0. Set $\eta_1 = \frac{1}{6}\delta_1$ and $\eta_{n+1} = \frac{1}{6}\min(\eta_n, \delta_{n+1})$ for $n \in \mathbb{N}$. By induction over $n \in \mathbb{N}$ one constructs $m_n \in \Gamma$ such that

$$\left(\sum_{i=1}^n (1 - \delta_i)|\alpha_i|\right) + \eta_n \sum_{i=1}^n |\alpha_i| \leq \left\| \sum_{i=1}^n \alpha_i x_{m_i} \right\| \leq \sum_{i=1}^n |\alpha_i| \quad \forall n \in \mathbb{N}, \alpha_i \in \mathbb{C}. \quad (15)$$

For the induction step $n \mapsto n+1$ fix an element $\alpha = (\alpha_i)_{i=1}^{n+1}$ in the unit sphere of l_{n+1}^1 such that $\alpha_{n+1} \neq 0$. Then (12) yields

$$\begin{aligned} & \left\| \left(\sum_{i=1}^n \alpha_i x_{m_i} \right) + \alpha_{n+1} x_m \right\| + \frac{1}{2} \min(\eta_n, \delta_{n+1}) \\ & \geq \left\| \sum_{i=1}^n \alpha_i x_{m_i} \right\| + |\alpha_{n+1}| \\ & \stackrel{(15)}{\geq} \left(\sum_{i=1}^n (1 - \delta_i)|\alpha_i| \right) + \eta_n \left(\sum_{i=1}^n |\alpha_i| \right) + |\alpha_{n+1}| \\ & \geq \left(\sum_{i=1}^{n+1} (1 - \delta_i)|\alpha_i| \right) + \min(\eta_n, \delta_{n+1}) \end{aligned}$$

whence

$$\left\| \left(\sum_{i=1}^n \alpha_i y_{m_i}^{(i)} \right) + \alpha_{n+1} y_m^{(n+1)} \right\| \geq \left(\sum_{i=1}^{n+1} (1 - \delta_i)|\alpha_i| \right) + 3\eta_{n+1}$$

for infinitely many m . This gives (15); the details are the same as in the proof of [23, Th. 2] or of Lemma 6. Note for the proof of part (b) below that we used the w^* -convergence of (x_n) only in order to apply property (m_1^*) but not for the construction of the l^1 -basis itself.

(ii) \Rightarrow (i): Suppose that Z satisfies (ii) without having (m_1^*) and suppose that Z' is L-embedded. Then there are $x \in X$, $\varepsilon > 0$ and a w^* -null sequence $(x_n) \subset X$ such that $\lim \|x + x_n\|$ and $\lim \|x_n\|$ exist and

$$\varepsilon + \lim \|x + x_n\| < \|x\| + \lim \|x_n\|. \quad (16)$$

Since (16) excludes the case $\|x_n\| \rightarrow 0$ we have - after passing to an appropriate subsequence - that $x_n \overset{\text{asy}}{\sim} l^1$. Let (x_{n_γ}) be a universal net. By Lemma 2 it w^* -converges to a point $x_s \in X_s$ and

$$\lim_{\gamma} \|x + x_{n_\gamma}\| \geq \|x + x_s\| = \|x\| + \lim_{\gamma} \|x_{n_\gamma}\|$$

by w^* -continuity of the norm which contradicts (16) and proves property (m_1^*) .

(ii) \Leftrightarrow (iii) is immediate from the definition of τ_μ .

(iii) \Rightarrow (iv): If Z is separable then B_X is w^* -sequentially compact. Hence B_X is τ_μ -sequentially compact if (iii) holds.

(iv) \Rightarrow (i): If Z is M-embedded then $Z = X^\#$. In this case, if B_X is τ_μ -sequentially compact then the proof of Proposition 17 shows that Z has (m_1^*) .

(b) The Schur property clearly implies (m_1) . Conversely, suppose a Banach space X has (m_1) but fails to have the Schur property. Then there exists a normalized w -null sequence (x_n) in X . Exactly as in (i) \Rightarrow (ii) of part (a), using (13) instead of (12), we can extract a subsequence (x_{n_k}) which spans l^1 asymptotically. But a normalized sequence spanning l^1 cannot converge weakly. This contradiction proves that (m_1) implies the Schur property. \blacksquare

Sublemma 19 *For any L-embedded space X one has the following inclusions*

$$\begin{aligned} X \cap \overline{B_{X_s}}^{w^*} & \subset \bigcap \{ \overline{V}^w \mid V \text{ } \tau_\mu\text{-open in } B_X, 0 \in V \} \\ & \subset \bigcap \{ \overline{c\omega}^{\|\cdot\|}(V) \mid V \text{ } \tau_\mu\text{-open in } B_X, 0 \in V \}. \end{aligned}$$

Proof: The first inclusion is immediate from (i) of Lemma 5, the second inclusion follows from $\overline{\text{co}}^w(V) = \overline{\text{co}}^{\|\cdot\|}(V)$. ■

That at the time of this writing we are not able to give the whole analogue of [10, L. 2.6] is essentially due to the fact that we know only the behaviour of sequences but not of nets in τ_μ , more specifically if we knew that one can extract a τ_μ -convergent sequence from a τ_μ -convergent net then we could also prove that $\bigcap \overline{\text{co}}^{\|\cdot\|}(V) \subset X \cap \overline{B_{X_s}}^{w*}$. In any case, our reduced version suffices for Lemma 20 which corresponds to [10, L. 2.7]. Note that (ii) of Lemma 20 is equivalent to the two conditions of Theorem 13.

Lemma 20 *Let X be an L -embedded Banach space. The following assertions are equivalent.*

- (i) *There exists a locally convex Hausdorff topology on X which is coarser than τ_μ on B_X .*
- (ii) *B_X is compact Hausdorff with respect to $\sigma(X, X^\#)$.*
- (iii) *$\{0\}$ is the intersection of the convex τ_μ -neighbourhoods of 0 in B_X .*

If moreover B_X is τ_μ -compact Hausdorff, then the above conditions are also equivalent to

- (iv) *The weak topology of X is finer than τ_μ on B_X .*
- (v) *There exists a locally convex Hausdorff topology - namely $\sigma(X, X^\#)$ - which coincides with τ_μ on B_X .*
- (vi) *0 has a basis of τ_μ -neighbourhoods in B_X consisting of convex sets.*

Proof: The proof is word-for-word the same as the one of [10, L. 2.7] (with Sublemma 19 replacing [10, L. 2.6]). ■

Suppose that with the notation of (i) of Lemma 5 one has $x = x'' \in X$; then $x_\gamma \xrightarrow{\tau_\mu} 0$ and $x_\gamma \xrightarrow{w} x$. If (iv) of Lemma 20 holds then this means that both $x_\gamma \xrightarrow{\tau_\mu} 0$ and $x_\gamma \xrightarrow{\tau_\mu} x$. Since it is not clear whether τ_μ is Hausdorff we cannot deduce that $x = 0$. Therefore we claimed B_X to be τ_μ -Hausdorff for (iv) - (vi) in Lemma 20 (in particular for the proof of (iv) \Rightarrow (ii)).

We end this section with some remarks on the continuity of the canonical L -projection P on the bidual of an L -embedded space X .

In [10, Prop. 3.8] it was shown that P is $(w^*-\tau_\mu)$ -sequentially continuous if and only if X has the Schur property. In the proof the authors of [10] use the Grothendieck property of L^∞ in order to obtain the (w^*-w) -sequential continuity of the L -projection on $(L^\infty)'$. But in the general case it is not known whether P is always (w^*-w) -sequentially continuous although by an example of Johnson it is known that in general the dual of an L -embedded Banach space does not have the Grothendieck property. Therefore it is not clear whether the first part of [10, Prop. 3.8] can be generalized in our context; its second part can:

Proposition 21 *Let X be L -embedded and P be the canonical L -projection on X'' . Suppose that the abstract measure topology τ_μ on X is Hausdorff. Then P is $(w^*-\tau_\mu)$ -continuous if and only if the restriction of τ_μ to B_X is compact Hausdorff locally convex.*

Proof: See [10, Prop. 3.8(b)]. ■

§6 The special case of the predual of a von Neumann algebra

Recall that two elements $\phi, \psi \in \mathcal{N}_*$ in the predual of a von Neumann algebra \mathcal{N} are called orthogonal - $\phi \perp \psi$ in symbols - if they have orthogonal right and orthogonal left support projections. It is well known that $\phi \perp \psi$ if and only if the linear span of ϕ and ψ is isometrically isomorphic to the two-dimensional $l^1(2)$.

Sublemma 22 *Let \mathcal{N} be a von Neumann algebra with predual $X = \mathcal{N}_*$. For every $\varepsilon > 0$ there is $\delta > 0$ with the following property.*

If $x, y, z \in B_X$ are such that

$$\begin{aligned} \left\| \alpha \frac{z}{\|z\|} + \beta \frac{x}{\|x\|} \right\| &\geq (1 - \delta)(|\alpha| + |\beta|) \\ \left\| \alpha \frac{z}{\|z\|} + \beta \frac{y}{\|y\|} \right\| &\geq (1 - \delta)(|\alpha| + |\beta|) \end{aligned}$$

then there are $\tilde{x}, \tilde{y}, \tilde{z} \in X$ of norm one such that

$$\begin{array}{l} \tilde{x} \perp \tilde{z} \\ \tilde{y} \perp \tilde{z} \end{array} \quad \text{and} \quad \begin{array}{l} \|x - \tilde{x}\| \leq \varepsilon \\ \|y - \tilde{y}\| \leq \varepsilon \\ \|z - \tilde{z}\| \leq \varepsilon \end{array}$$

Proof. We have already recalled the fact that two normal functionals on a von Neumann algebra are orthogonal if and only if they span $l^1(2)$ isometrically. A second ingredient of this proof is the fact that the ultraproduct of a family of preduals of von Neumann algebras is again a predual of a von Neumann algebra [12], [27]. It is now enough to combine these two facts with a standard ultraproduct argument [14]. ■

It is well known that if \mathcal{N} is a von Neumann algebra with a finite faithful normal trace τ then the measure topology on the predual \mathcal{N}_* defined via τ is metrizable and makes \mathcal{N}_* a Hausdorff topological vector space. For the abstract measure topology τ_μ on the predual of an arbitrary von Neumann algebra we only know by the following lemma (and by Lemmas 5, 6) that multiplication by scalars and addition are continuous and that on bounded sets τ_μ makes the von Neumann predual a Fréchet space.

Proposition 23 *Let $X = \mathcal{N}_*$ be the predual of a von Neumann algebra \mathcal{N} . If X is endowed with the abstract measure topology τ_μ which it has by Theorem 4 then addition is τ_μ -continuous.*

Proof. Let $(x_n), (y_n)$ be two sequences in B_X each of which spans l^1 asymptotically. Suppose there is $\varepsilon > 0$ such that $\|z_n\| > \varepsilon$ where $z_n = x_n + y_n$, suppose further that $\lim \|x_n\|$, $\lim \|y_n\|$, $\lim \|z_n\|$ exist. To show the Proposition it is enough to show that there is a subsequence (z_{n_k}) spanning l^1 asymptotically.

Let (δ_n) be a sequence of strictly positive numbers in $]0, 1]$ converging to 0. Set $\eta_1 = \frac{1}{3}\delta_1$ and $\eta_{n+1} = \frac{1}{3} \min(\eta_n, \delta_{n+1})$ for $n \in \mathbb{N}$. By induction over $m \in \mathbb{N}$ we construct a sequence (n_m) such that

$$\left\| \sum_{k=1}^m \alpha_k \frac{z_{n_k}}{\|z_{n_k}\|} \right\| \geq \left(\sum_{k=1}^m (1 - \delta_k) |\alpha_k| \right) + \eta_m \sum_{k=1}^m |\alpha_k| \quad (17)$$

for all integers m and all scalars α_k . Since the construction resembles the proofs of [23, Th. 2], of Lemma 6, and of Remark 18 we detail it only for $m = 2$.

Let $\theta_2 > 0$ be arbitrary for the moment and to be determined later. Set $n_1 = 1$.

CLAIM: There is $n_2 \in \mathbb{N}$ such that

$$\left\| \alpha \frac{z_1}{\|z_1\|} + \beta \frac{x_{n_2}}{\|x_{n_2}\|} \right\| \geq (1 - \theta_2) (|\alpha| + |\beta|) \quad (18)$$

$$\left\| \alpha \frac{z_1}{\|z_1\|} + \beta \frac{y_{n_2}}{\|y_{n_2}\|} \right\| \geq (1 - \theta_2) (|\alpha| + |\beta|) \quad (19)$$

for all scalars α, β . (As usual in this paper) this is essentially due to the w^* -lower semicontinuity of the norm by which one has $\liminf_\gamma \left\| \alpha \frac{z_1}{\|z_1\|} + \beta \frac{x_{n_\gamma}}{\|x_{n_\gamma}\|} \right\| \geq \left\| \alpha \frac{z_1}{\|z_1\|} + \beta \frac{x_s}{\|x_s\|} \right\| = |\alpha| + |\beta|$ where $x_s \in X_s$ is the w^* -limit of a universal net (x_{n_γ}) and $\|x_s\| = \lim_\gamma \|x_{n_\gamma}\|$ (see Lemma 2); hence there are infinitely many n_2 satisfying (18). Applying the same reasoning to the corresponding subsequence of (y_n) gives the CLAIM.

By Sublemma 22 there are $\tilde{x}_2, \tilde{y}_2, \tilde{z}_1$ in X such that

$$\|x_{n_2} - \tilde{x}_2\| \leq \theta'_2, \quad \|y_{n_2} - \tilde{y}_2\| \leq \theta'_2, \quad \|z_1 - \tilde{z}_1\| \leq \theta'_2,$$

and

$$\tilde{z}_1 \perp \tilde{x}_2, \quad \tilde{z}_1 \perp \tilde{y}_2 \quad (20)$$

where $\theta'_2 \rightarrow 0$ as $\theta_2 \rightarrow 0$. Now (20) implies that

$$\tilde{z}_1 \perp \tilde{z}_{n_2}$$

where $\tilde{z}_{n_2} = \tilde{x}_{n_2} + \tilde{y}_{n_2}$ which means that $\left\| \alpha \frac{\tilde{z}_1}{\|\tilde{z}_1\|} + \beta \frac{\tilde{z}_2}{\|\tilde{z}_2\|} \right\| = |\alpha| + |\beta|$. Hence, if θ_2 is chosen small enough, we get (17) for $m = 2$. It is now left to the reader to iterate the construction in order to end the induction and thus the proof. ■

As already noticed in the remarks concerning the questions after Corollary 9, Randrianantoanina [26] has proved that the Kadec-Pelczyński splitting lemma holds for preduals of von Neumann algebras. With this result Komlos' theorem follows almost immediately for von Neumann preduals.

Proposition 24 (Komlos) *Each bounded sequence in the predual of a von Neumann algebra admits a subsequence whose Cesaro means converge with respect to the abstract measure topology.*

Proof: Let $X = \mathcal{N}_*$ be the predual of a von Neumann algebra \mathcal{N} . Endow X with its abstract measure topology τ_μ . Let $(x_n) \subset X$ be bounded. Then by [26] there is a subsequence (x_{n_k}) and there is a decomposition $x_{n_k} = y_k + z_k$ where (z_k) w -converges and such that there is a sequence (\tilde{y}_k) of pairwise orthogonal elements in X with

$$\|\tilde{y}_k - y_k\| \rightarrow 0. \quad (21)$$

If one applies the classical theorem of Komlos to the isometric l^1 -copy $[\tilde{y}_k]$ then one obtains, after passing to an appropriate subsequence, that the Cesaro means of any subsequence of (\tilde{y}_k) converge with respect to the measure (=pointwise) topology of $[\tilde{y}_k]$ whence with respect to τ_μ . So do the Cesaro means \hat{y}_k of the y_k by (21). It is known that von Neumann preduals have the weak Banach-Saks property [2]. Hence, again after passing to the appropriate subsequences of (y_k) and (z_k) , the Cesaro means \hat{z}_k of the z_k converge in norm whence with respect to τ_μ . Since $\hat{x}_k = \hat{y}_k + \hat{z}_k$ where \hat{x}_k denote the Cesaro means of the x_{n_k} , the assertion now follows from Proposition 23. ■

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